Tenth Edition

Elementary and Middle School MATHEMATICS

Teaching Developmentally

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TENTH EDITION

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Text Font: Janson Text LT Pro 9.75/12

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Library of Congress Cataloging-in-Publication Data

Names: Van de Walle, John A. | Karp, Karen, 1951- author. | Bay-Williams, Jennifer M., author. | Wray, Jonathan A., author.
Title: Elementary and middle school mathematics : teaching developmentally.
Description: Tenth edition / John A. Van de Walle, late of Virginia Commonwealth University, Karen S. Karp, University of Louisville, Jennifer M. Bay-Williams, University of Louisville ; with contributions by Jonathan Wray, Howard County Public Schools. | Boston : Pearson, [2019] | Includes bibliographical references and index.
Identifiers: LCCN 2017037185 | ISBN 9780134802084
Subjects: LCSH: Mathematics--Study and teaching (Elementary) | Mathematics--Study and teaching (Middle school)
Classification: LCC QA135.6 .V36 2019 | DDC 510.71/2--dc23
LC record available at https://lccn.loc.gov/2017037185

10 9 8 7 6 5 4 3 2 1

Pearson

ISBN 10: 0-13-480208-X
ISBN 13: 978-0-13-480208-4

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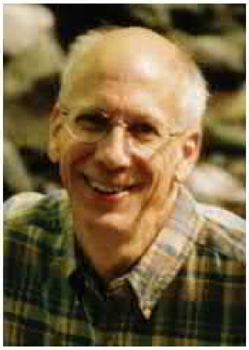


John A. Van de Walle

The late John A. Van de Walle was a professor emeritus at Virginia Commonwealth University. He was a leader in mathematics education who regularly gave professional development workshops for K–8 teachers in the United States and Canada focused on mathematics instruction that engaged students in mathematical reasoning and problem solving. He visited and taught in many classrooms and worked with teachers to implement student-centered mathematics lessons. He co-authored the *Scott Foresman-Addison Wesley Mathematics K–6* series and contributed to the original Pearson School mathematics program *enVisionMATH*. Additionally, John was very active in the National Council of Teachers of Mathematics (NCTM), writing book chapters and journal articles, serving on the board of directors, chairing the educational materials committee, and speaking at national and regional meetings.

Karen S. Karp

Karen S. Karp is a visiting professor at Johns Hopkins University (Maryland). Previously, she was a professor of mathematics education at the University of Louisville for more than twenty years. Prior to entering the field of teacher education, she was an elementary school teacher in New York. She is the coauthor of *Developing Essential Understanding of Addition and Subtraction for Teaching Mathematics in PreK–Grade 2*, *Discovering Lessons for the Common Core State Standards in Grades K–5* and *Putting Essential Understanding of Addition and Subtraction into Practice PreK–Grade 2*. She is a former member of the board of directors for the National Council of Teachers of Mathematics (NCTM) and a former president of the Association of Mathematics Teacher Educators. She continues to work in classrooms to support teachers in ways to instruct students with disabilities.

Jennifer M. Bay-Williams

Dr. Jennifer Bay-Williams is a professor at the University of Louisville. She has written many articles and books around K–12 mathematics education, including the three book series related to this book—*Teaching Student Centered Mathematics*—and various other books, including *Everything You Need for Mathematics Coaching*, *Developing Essential Understanding of Addition and Subtraction in Prekindergarten–Grade 2*, *On the Money* (financial literacy book series), and *Math and Literature: Grades 6–8*. Jennifer taught elementary, middle, and high school in Missouri and in Peru, and continues to learn and work in K–8 classrooms in rural and urban settings. Jennifer has been a member of the NCTM board of directors, AMTE secretary, president, and lead writer for *Standards for the Preparation of Teachers of Mathematics* (AMTE, 2017), and is currently on the TODOS: Mathematics for All Board of Directors.

About the Contributors

Jonathan Wray is the technology contributor to *Elementary and Middle School Mathematics, Teaching Developmentally* (sixth–tenth editions). He is the acting coordinator of secondary mathematics curricular programs in the Howard County public school system. He has served as the president of the Association of Maryland Mathematics Teacher Educators (AMMTE) and the Maryland Council of Teachers of Mathematics (MCTM) and currently is manager of the Elementary Mathematics Specialists and Teacher Leaders (ems&tl) Project. Jon also served on the NCTM Board of Directors (2012–2015). He has been recognized for his expertise in infusing technology in mathematics teaching and was named an outstanding technology leader in education by the Maryland Society for Educational Technology (MSET). He was a primary and intermediate grades classroom teacher, gifted and talented resource teacher, elementary mathematics specialist, curriculum and assessment developer, grant project manager, and educational consultant.

Elizabeth Todd Brown is the assessment and media contributor to *Elementary and Middle School Mathematics, Teaching Developmentally* (tenth edition). She is a professor emeritus from the University of Louisville. She was also an elementary and middle school teacher in Iowa, Missouri, and Kentucky and continues to volunteer weekly at a local elementary school in Louisville. Todd coauthored the book, *Feisty Females*, and has written articles on the importance of early mathematics learning. She served as president of the local NCTM affiliate Greater Louisville Council of the Teachers of Mathematics and was the chair of the volunteers for the NCTM Regional in Louisville in 1998 and 2013. Todd was the 1998 Kentucky Presidential Awardee for Excellence in Teaching Mathematics and Science.

Brief Contents

Contents

PART I Teaching Mathematics: Foundations and Perspectives

The fundamental core of effective teaching of mathematics combines an understanding of how students learn, how to promote that learning by teaching through problem solving, and how to plan for and assess that learning daily. That is the focus of these first six chapters, providing discussion, examples, and activities that develop the core ideas of learning, teaching, planning, and assessment for each and every student.

PART II Teaching Student-Centered Mathematics

Each of these chapters *applies* the core ideas of Part I to the content taught in K–8 mathematics. Clear discussions are provided for how to teach the topic, what a learning progression for that topic might be, and what worthwhile tasks look like. Hundreds of problem-based, engaging tasks and activities are provided to show how the concepts can be developed with students. These chapters are designed to help you develop pedagogical strategies now, and serve as a resource and reference for your teaching now and in the future.

Preface

All students can learn mathematics with understanding. It is through the teacher's actions that every student can have this experience. We believe that teachers must create a classroom environment in which students are given opportunities to solve problems and work together, using their ideas and strategies, to solve them. Effective mathematics instruction involves posing tasks that engage students in the mathematics they are expected to learn. Then, by allowing students to interact with and productively struggle with *their own mathematical ideas* and *their own strategies*, they will learn to see the connections among mathematical topics and the real world. Students value mathematics and feel empowered to use it.

Creating a classroom in which students design solution pathways, engage in productive struggle, and connect one mathematical idea to another, is complex. Questions arise, such as, "How do I get students to wrestle with problems if they just want me to show them how to do it? What kinds of tasks lend themselves to this type of engagement? Where can I learn the mathematics content I need to be able to teach in this way?" With these and other questions firmly in mind, we have several objectives in the tenth edition of this textbook:

1. Illustrate what it means to teach mathematics using a problem-based approach.
2. Serve as a go-to reference for all of the mathematics content suggested for grades preK–8 as recommended in the Common Core State Standards (NGA Center & CCSSO, 2010) and in standards used in other states, and for the research-based strategies that illustrate how students best learn this content.
3. Present a practical resource of robust, problem-based activities and tasks that can engage students in the use of significant mathematical concepts and skills.
4. Focus attention on student thinking, including the ways students might reason about numbers, and possible challenges and misconceptions they might have.

We are hopeful that you will find that this book is a valuable resource for teaching and learning mathematics!

New to this Edition

The following are highlights of the most significant changes in the tenth edition.

Common Challenges and Misconceptions

Every chapter in Part II offers at least one table that summarizes common challenges students encounter in learning that topic (Chapter 15, Fraction Operations has three). The table includes the challenge, provides an example of what that might look like in either a sample of student work or a statement, and then offers some brief ideas of what you might do to help. Knowing common student challenges and misconceptions is a critical part of planning and can greatly influence how a lesson is structured and what problems you use. The research from many sources has been merged into these practical references.

Routines

More and more classrooms are using innovative lesson designs and short discussion routines to help students develop number sense, flexibility, and the mathematical practices. In Chapter 4, we have added se new sections on: 3-Act Tasks, Number Talks, and Worked Examples. For example, worked examples are mentioned in some of the tables identifying student challenges, because there is research to suggest that analyzing worked examples is effective in helping students learn.

Mathematical Modeling

Since the ninth edition, there has been significant national dialogue about the importance of mathematical modeling and what this might look like across the grades. The *Guidelines for Assessment & Instruction in Mathematical Modeling Education* (GAIMME) Report (COMAP & SIAM, 2016) provides excellent guidance. Therefore, the section in Chapter 13 on mathematical modeling was completely rewritten to reflect the GAIMME report, as well as to showcase a number of excellent books and articles that have emerged recently.

Infusion of Technology

You may notice that Chapter 7 (Technology) from the previous edition is gone. Readers and reviewers have commented that this chapter is not needed in part because using technology is much more commonly understood and used, and in part because it makes far more sense to talk about technology *as it relates to the mathematics.* We have heard you and we have integrated technology discussions, tools, and ideas throughout the book.

MyLab Education

Digital learning and assessment resources have been expanded significantly via MyLab Education. The following resources have been designed to help you develop the pedagogical knowledge *and* content knowledge needed to be a successful teacher of mathematics:

- **Video examples:** Embedded throughout all chapters, these examples allow you to see key concepts in action through authentic classroom video, as well as clips of children solving math problems. Additional videos feature your authors and other experts introducing and briefly explaining strategies for teaching important topics.
- **Self-checks:** Designed for self-study, these multiple-choice items are tied to each chapter learning outcome, and help you assess how well you have mastered the concepts covered in the reading. These exercises are self-grading and provide a rationale for the correct answer.
- **Application exercises:** Video and scenario-based exercises appear throughout the chapters and provide an opportunity for you to apply what you have learned to real classroom situations. There are also ten exercises on *observing and responding to student thinking* that include video clips of children talking through and solving problems on a whiteboard app; accompanying questions ask you to analyze and child's reasoning, identify any misconceptions, and explain any actions or prompts you might use as the teacher to guide the student's learning. Expert feedback is provided after submitting your response.
- **Math practice:** Located at the end of most content chapters, these sets of questions provide an opportunity to practice or refresh your own mathematics skills through solving exercises associated with the content from that chapter. These questions are also self-grading.
- **Blackline masters, activity pages, and expanded lessons:** These documents are linked throughout each chapter and make it easy for instructors and students to download and print classroom-ready handouts that can be used in a methods class or school settings.

Major Changes to Specific Chapters

Every chapter in the tenth edition has been revised to reflect the most current research, standards, and exemplars. This is evident in the approximately *300 new references* in the tenth edition! This represents our ongoing commitment to synthesize and present the most current evidence of effective mathematics teaching. Here we share changes to what we consider the most significant (and that have not already been mentioned above).

Teaching Mathematics in the 21st Century (Chapter 1)

The new Association of Mathematics Teacher Educators (AMTE) Standards for Preparing Teachers of Mathematics (AMTE, 2017) are described in Chapter 1. We added a section on how to create a whole school agreement with a cohesive mathematics message.

Exploring What It Means to Know and Do Mathematics (Chapter 2)

Chapter 2 was revised in several significant ways, including revisions to the exemplar tasks (one in each content domain) to each have a common format, and to each have a stronger focus on multiple strategies. The discussions on theory were condensed, and making connections between theory and teaching were revised to be more succinct and explicit.

Teaching through Problem Solving (Chapter 3)

The NCTM Teaching Practices (2014) have been integrated into Chapter 3. A completely revamped section, now titled Developing Procedural Fluency, focuses on the importance of connecting conceptual and procedural knowledge, and includes a new list of ways to adapt drill-related tasks to emphasize understanding and connections (Boaler, 2016). Talk moves in the Discourse section have been revised to include eight talk moves (Chapin, O'Conner, & Anderson, 2013).

Teaching through Problem Solving (Chapter 4)

Beyond the new routines section (described above), the families section was heavily revised and the lesson plan steps condensed and formatted for easier readability.

Teaching Mathematics Equitably to All Students (Chapter 6)

We expanded our emphasis on using an asset-based approach, focusing on students' strengths rather than deficits. We emphasize a focus on using students' prior knowledge and experiences to drive instructional decisions. There is also a revamping of the section on gifted and talented students including attention to an excellence gap (students who may be overlooked).

Basic Facts (Chapter 9)

Recent research (e.g., Baroody et al., 2016) has uncovered a new and effective addition reasoning strategy—Use 10, which has been added to this chapter, along with new visuals and insights on teaching subtraction facts effectively.

Developing Strategies for Multiplication and Division (Chapter 12)

In new updates in this chapter, there are expanded examinations of the written records of computing multiplication and division problems including lattice multiplication, open arrays, and partial quotients. There is also a new section of the use of the break apart or decomposition strategy for division. A conversation about the selection of numbers for computational estimation problems is also shared.

Algebraic Thinking, Equations, and Functions (Chapter 13)

In addition to the new section on mathematical modeling, there are several new ideas and strategies for supporting algebraic thinking, including adapting the hundreds chart to explore patterns and options for creating tables with more structure to help students notice relationships.

Developing Fraction Concepts (Chapters 14)

Fraction concepts has an expanded focus on the fundamental ideas of sharing and iterating. This chapter also has been reorganized, has more contexts for comparing fractions, and more attention to student challenges in understanding fractions.

Ratios, Proportions, and Proportional Reasoning (Chapter 17)

The sections on additive and multiplicative reasoning have been significantly revised, including a new discussion on social justice mathematics. Additionally, significantly more literature connections are provided in this chapter and new activities.

Developing Concepts of Data Analysis (Chapter 20)

This chapter had numerous enhancements and changes! In addition to four new figures and completely updated technology options, the discussion of variability is woven throughout the chapter (including more attention to measures that are resistant to outlier), and sections on boxplots, histograms, and bivariate data were expanded and revised (see new subsection on bivariate categorical data).

An Introduction to *Teaching Developmentally*

If you look at the table of contents, you will see that the chapters are separated into two distinct sections. The first section consists of six chapters and covers important ideas that cross the boundaries of specific areas of content. The second section, consisting of 16 chapters, offers teaching suggestions and activities for every major mathematics topic in the preK–8 curriculum. Chapters in Part I offer perspectives on the challenging task of helping students learn mathematics. Having a feel for the discipline of mathematics—that is, to know what it means to "do mathematics"—is critical to learning how to teach mathematics well. In addition, understanding constructivist and sociocultural perspectives on learning mathematics and how they are applied to teaching through problem solving provides a foundation and rationale for how to teach and assess preK–8 students. You will be teaching diverse students including students who are English learners, are gifted, or have disabilities. In this text, you will learn how to apply instructional strategies in ways that support and challenge *all* learners. Formative assessment strategies and strategies for diverse learners are addressed in specific chapters in Part I (Chapters 5, and 6, respectively), and throughout Part II chapters.

Each chapter of Part II focuses on one of the major content areas in preK–8 mathematics curriculum. It begins with identifying the big ideas for that content, and provides guidance on how students best learn that content through many problem-based activities to engage them in understanding mathematics, as well as considering what challenges they may encounter and how you might help them.

Hundreds of tasks and activities are embedded in the text. Take out pencil and paper, or use technology, and try the problems, thinking about how you might solve them *and* how students at the intended grades might solve them. This is one way to actively engage in *your learning* about *students learning* mathematics. In so doing, this book will increase your own understanding of mathematics, the students you teach, and how to teach them effectively.

Some Special Features of This Text

By flipping through the book, you will notice many section headings, a large number of figures, and various special features. All are designed to make the book more useful as a long-term resource. Here are a few things to look for.

CHAPTER 14

Developing Fraction Concepts

LEARNER OUTCOMES

After reading this chapter and engaging in the embedded activities and reflections, you should be able to:

14.1 Describe and give examples for fractions constructs and fraction models.

14.2 Explain foundational concepts of fractional parts, including iteration and partitioning.

14.3 Illustrate the concept of equivalence across fraction models.

14.4 Describe strategies for comparing fractions and ways to teach this topic conceptually.

Fractions are one of the most important topics students need to understand to be successful in algebra and beyond, yet it is an area in which U.S. students, as well as students in many countries, struggle (OECD, 2014). National Assessment of Educational Progress (NAEP) results have consistently shown that students have a weak understanding of fraction concepts (Sowder & Wearne, 2006; Wearne & Kouba, 2000). This lack of understanding is then translated into difficulties with fraction computation, decimal and percent concepts, and algebra (Bailey, Hoard, Nugent, & Geary, 2012; Booth & Newton, 2012; Brown & Quinn, 2007; National Mathematics Advisory Panel, 2008; Siegler, Fazio, Bailey, & Zhou, 2013). Therefore, it is critical that you teach fractions well, present fractions as interesting and important, and commit to helping students understand the meaning of fractions.

BIG IDEAS

- Fractions can and should be represented across different interpretations (e.g., part-whole and division) and different models: area (e.g., [illegible] of a garden), length (e.g., [illegible] of an inch), and set (e.g., [illegible] of the marbles).
- Fractions are equal shares of a whole or a unit. Therefore, equal sharing activities (e.g., 2 sandwiches shared with 4 friends) build on whole-number knowledge to introduce fractional quantities.
- Partitioning and iterating are ways students to understand the meaning of fractions. Partitioning can be thought of as splitting the whole equally (e.g., splitting a whole into fourths), and iterating can be thought of as making a copy of each piece and counting them (e.g., one-fourth, two-fourths, etc.).
- Equivalent fractions are ways of describing the same amount by using different-sized fractional parts.
- Fractions can be compared by reasoning about the relative size of the fractions. Estimation and reasoning are important in teaching understanding of fractions.

337

◀ Learning Outcomes

To help readers know what they should expect to learn, each chapter begins with learning outcomes. Self-checks are numbered to cover and thus align with each learning outcome.

◀ Big Ideas

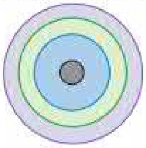
Much of the research and literature espousing a student-centered approach suggests that teachers plan their instruction around big ideas rather than isolated skills or concepts. At the beginning of each chapter in Part II, you will find a list of the big mathematical ideas associated with the chapter. Teachers find these lists helpful to quickly envision the mathematics they are to teach.

Activity 21.9

CCSS-M: 7.G.B.4; 7.SP.C.6; 7.SP.C.7b

Chance of Hitting the Target?

Project a target such as the one illustrated here with concentric circles having radii of 2 inches, 6 inches, 8 inches, and 10 inches, each region shaded a different color. Ask students to determine the fraction and percent of each colored region in the circle.



Ask students to discuss what the probability for landing on the center (assuming all throws land on the circle and are thrown randomly). Ask students to discuss why data may or may not match the percent of the area that is covered (e.g., people with good aim will be able to hit the smaller areas more often). Then, have students propose what point values they would assign to each region. Students may assign values in various ways. For example, they may think the skinny outer circle is harder to land on and give it more points than other sections, even though the area of that region may be more. Allow them time to share their reasoning and to critique others' ways of assigning points.

◀ Activities

The numerous activities found in every chapter of Part II have always been rated by readers as one of the most valuable parts of the book. Some activity ideas are described directly in the text and in the illustrations. Others are presented in the numbered Activity boxes. Every activity is a problem-based task (as described in Chapter 3) and is designed to engage students in doing mathematics.

Adaptations for Students with Special Needs and English Learners ▶

Chapter 6 provides detailed background and strategies for how to support students with special needs and English learners (ELs). But, many adaptations are specific to a activity or task. Therefore, Part II chapters offer adaptations and instructions within activities (look for the icon) that can meet the needs of students with special needs and ELs.

Activity 9.2

CCSS-M: 1.OA.A.1; 1.OA.C.6; 2.OA.B.2

How Many Feet in the Bed?

ENGLISH LEARNERS

STUDENTS with SPECIAL NEEDS

Read *How Many Feet in the Bed?* by Diane Johnston Hamm. On the second time through the book, ask students how many more feet are in the bed when a new person gets in. Ask students to record the equation (e.g., 6 + 2) and tell how many. Two less can be considered as family members get out of the bed. Find opportunities to make the connection between counting on and adding using a number line. For ELs, be sure that they know what the phrases "two more" and "two less" mean (and clarify the meaning of *foot*, which is also used for measuring). Acting out with students in the classroom can be a great illustration for both ELs and students with disabilities.

FORMATIVE ASSESSMENT Notes. To assess understanding of division algorithms, call on different students to explain individual steps using the appropriate terminology that connects to the concept of division. Use an Observation Checklist to record students' responses, indicating how well they understand the algorithm. For students who are having difficulty, you may want to conduct a short diagnostic interview to explore their level of understanding in more detail. Begin by having the student complete $115 \div 9$ and ask them to talk about what they are thinking as they carry out specific steps in the process. If there is difficulty explaining, have the student use base-ten materials to directly model the problem and attempt to link the actions to the procedure. Then ask them to discuss verbally the connections between what was done with the models and what was written symbolically. ■

◀ Formative Assessment Notes

Assessment is an integral process within instruction. Similarly, it makes sense to think about what to be listening for (assessing) as you read about different areas of content development. Throughout the content chapters, there are formative assessment notes with brief descriptions of ways to assess the topic in that section. Reading these assessment notes as you read the text can help you understand how best to assist students who struggle.

Technology Notes ▶

Infusing technological tools is important in learning mathematics. We have infused technology notes throughout Part II. A technology icon is used to identify places within the text or activity where a technology idea or resource is discussed. Descriptions include open-source (free) software, applets, and other Web-based resources, as well as ideas for calculator use.

TECHNOLOGY Note. An amazing computer tool for drawing two-dimensional views of block buildings is the Isometric Drawing Tool, available at the NCTM Illuminations website. Using mouse clicks students can draw either whole cubes, faces, or just lines. The drawings, however, are actually "buildings" and can be viewed as three-dimensional objects that when rotated can be seen from any vantage point. Prepared investigations lead students through the features of the tool. ■

CCSS Standards for Mathematical Practice

MP1. Make sense of problems and persevere in solving them.

◀ Standards for Mathematical Practice Margin Notes

Connections to the eight Standards of Mathematical Practice from the *Common Core State Standards* are highlighted in the margins. The location of the note indicates an example of the identified practice in the nearby text.

RESOURCES FOR CHAPTER 14

LITERATURE CONNECTIONS

The Doorbell Rang
Hutchins (1986)
Often used to investigate whole-number operations of multiplication and division, this book is also an excellent early introduction to fractions. The story is a simple tale of two children preparing to share a plate of 12 cookies. Just as they have figured out how to share the cookies, the doorbell rings and more children arrive. You can change the number of children to create a sharing situation that requires fractions (e.g., 8 children).

The Man Who Counted: A Collection of Mathematical Adventures
Tahan (1993)
This book contains a story, "Beasts of Burden," about a wise mathematician, Beremiz, and the narrator, who are traveling together on one camel. They are asked by three brothers to solve an argument: *Their father has left them 35 camels to divide among them: half to one brother, one-third to another, and one-ninth to the third brother.* The story is an excellent context for fractional parts of sets (and adding fractions). Changing the number of camels to 36 or 34, does not solve the challenge because the sum of $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{9}$ will never be one whole, no matter how many camels are involved. See Bresser (1995) for three days of activities with this book.

Apple Fractions
Pallotta (2002)
This book offers interesting facts about apples while introducing fractions as fair shares (of apples, a healthier option than books that focus on chocolate and cookies!). In addition, the words for fractions are used and connected to fraction symbols, making it a good connection for fractions in grades 1–3.

RECOMMENDED READINGS

Articles

Clarke, D. M., Roche, A., & Mitchell, A. (2008). Ten practical tips for making fractions come alive and make sense. *Mathematics Teaching in the Middle School, 13*(7), 373–380.
Ten excellent tips for teaching fractions are discussed and favorite activities are shared. An excellent overview of teaching fractions.

Lewis, R. M., Gibbons, L. K., Kazemi, E., & Lind T. (2015). Unwrapping students ideas about fractions. *Teaching Children Mathematics, 22*(3), 158–168.
This excellent read provides a how-to for implementing sharing tasks, including sequencing of tasks, questions to pose, and formatives assessment tool to monitor student understanding.

Freeman, D. W., & Jorgensen, T. A. (2015). Moving beyond brownies and pizzas. *Teaching Children Mathematics, 21*(7), 412–420.
This article describes student thinking as they compare fractions. In the more4U pages, they offer excellent sets of tasks with a range of contexts, each set focusing on a different reasoning strategy for comparing fractions.

Books

Lamon, S. (2012). *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies.* New York, NY: Taylor & Francis Group.
As the title implies, this book has a wealth of information to help with better understanding fractions and teaching fractions well. Many rich tasks and student work are provided throughout.

McNamara, J., & Shaughnessy, M. M. (2010). *Beyond pizzas and pies: 10 essential strategies for supporting fraction sense (grades 3–5).* Sausalito, CA: Math Solutions Publications.
This book has it all—classroom vignettes, discussion of research on teaching fractions, and many activities, including student work.

Websites

Rational Number Project (http://www.cehd.umn.edu/ci/rationalnumberproject/rnp1-09.html).
This project offers excellent lessons and other materials for teaching fraction concepts effectively.

◀ End of Chapter Resources

The end of each chapter there are *Resources*, which include "Literature Connections" (found in all Part II chapters) and "Recommended Readings."

Literature Connections. Here you will find examples of great children's literature for launching into the mathematics concepts in the chapter just read. For each title suggested, there is a brief description of how the mathematics concepts in the chapter can be connected to the story. These literature-based mathematics activities will help you engage students in interesting contexts for doing mathematics.

Recommended Readings. In this section, you will find an annotated list of articles and books to augment the information found in the chapter. These recommendations include NCTM articles and books, and other professional resources designed for the classroom teacher.

Supplements for Instructors

Qualified college adopters can contact their Pearson sales representatives for information on ordering any of the supplements described below. These instructor supplements are all posted and available for download (click on Educators) from the Pearson Instructor Resource Center at www.pearsonhighered.com/irc. The IRC houses the following:

- **Instructor's resource manual:** The Instructor's Resource Manual for the tenth edition includes a wealth of resources designed to help instructors teach the course, including chapter notes, activity suggestions, and suggested assessment and test questions.
- **Electronic test bank:** An electronic test bank (TB) contains hundreds of challenging questions as multiple-choice or short-answer questions. Instructors can choose from these questions and create their own customized exams.
- **PowerPoint™ presentation:** Ideal for instructors to use for lecture presentations or student handouts, the PowerPoint presentation provides ready-to-use graphics and text images tied to the individual chapters and content development of the text.

Acknowledgments

Many talented people have contributed to the success of this book and we are deeply grateful to all those who have assisted over the years. Without the success of the first edition, there would certainly not have been a second, much less ten editions. The following people worked closely with John on the first edition, and he was sincerely indebted to Warren Crown, John Dossey, Bob Gilbert, and Steven Willoughby, who gave time and great care in offering detailed comments on the original manuscript.

In preparing this tenth edition, we have received thoughtful input from the following mathematics teacher educators who offered comments on the ninth edition. Each reviewer challenged us to think through important issues. Many specific suggestions have found their way into this book, and their feedback helped us focus on important ideas. Thank you to Jessica Cohen, Western Washington University; Shea Mosely Culpepper, University of Houston; Shirley Dissler, High Point University; Cynthia Gautreau, California State University in Fullerton; Kevin LoPresto, Radford University; Ryan Nivens, East Tennessee State University; Adrienne Redmond-Sanogo, Oklahoma State University; and Douglas Roebuck, Ball State University. We are indebted to you for your dedicated and professional insight.

Additionally, we are very grateful for the ideas and reviews as we developed the tenth edition. Graham Fletcher, mathematics specialist, Atlanta, Georgia, provided strong support in the development of high leverage routines, including, of course, the new section on 3-Act Tasks. Susan Peters, mathematics teacher educator, University of Louisville, provided critical feedback and helpful ideas for developing concepts of data analysis. Their input resulted in significant improvements to those chapters. We continue to seek suggestions from teachers who use this book so please email us at teachingdevelopmentally@gmail.com with any advice, ideas, or insights you would like to share.

We are extremely grateful to our Pearson team of editors! Each of them has worked hard to turn our ideas (and yours) into a reality. And that is why we have been able to continue to evolve this book in a way to make it accessible online and via hard copy. Drew Bennett, our editor, has helped us define the direction of this edition and make important decisions that would make the book a better product for pre-service and in-service teachers. Our development editor, Kim Norbuta, has been supportive and positive, keeping us on target, even with the tightest of deadlines. Our content producer Yagnesh Jani was always available with the missing resources and answers we needed. Finally, we are very grateful to Jason Hammond and his editing team at SPi-Global, who carefully and conscientiously assisted in preparing this edition for publication. It has been a pleasure to interact with each of them and they have given us peace of mind to have knowledgeable, strong support.

We would each like to thank our families for their many contributions and support. On behalf of John, we thank his wife, Sharon, who was John's biggest supporter and a sounding board as he wrote the first six editions of this book. We also recognize his daughters, Bridget (a fifth-grade teacher in Chesterfield County, Virginia) and Gretchen (an Associate Professor of psychology at Rutgers University–Newark). They were John's first students, and he tested many ideas that are in this book by their sides. We can't forget those who called John "Math Grandpa": his granddaughters, Maggie, Aidan, and Grace.

From Karen Karp: I would like to express thanks to my husband, Bob Ronau, who as a mathematics educator graciously helped me think about decisions while offering insights and encouragement. In addition, I thank my children, Matthew, Tammy, Joshua, Misty, Matt, Christine, Jeffrey, and Pamela for their kind support and inspiration. I also am grateful for my wonderful grandchildren, Jessica, Zane, Madeline, Jack and Emma, who have helped deepen my understanding about how children think.

From Jennifer Bay-Williams: I would like to begin by saying thank you to the many mathematics teachers and teacher educators whose presentations at conferences, blogs, tweets, articles and classroom lessons have challenged and inspired me. I am forever grateful to my husband, Mitch Williams, whose background in English/Language Arts and great listening skills have been an amazing support. Finally, thank you to my children, MacKenna (14 years) and Nicolas (11 years), along with their peers and teachers, who continue to help me think more deeply about mathematics teaching and learning.

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CHAPTER 1

Teaching Mathematics in the 21st Century

LEARNER OUTCOMES

After reading this chapter and engaging in the embedded activities and reflections, you should be able to:

1.1 Summarize the factors that influence the effective teaching of mathematics.

1.2 Describe the importance of content standards, process standards and standards of mathematical practice.

1.3 Explore the qualities needed to learn and grow as a professional teacher of mathematics.

Some of you will soon find yourself in front of a class of students; others of you may already be teaching. What general ideas will guide the way you will teach mathematics as you grow in the teaching profession? This book will help you become comfortable with the mathematics content of the preK–8 curriculum. You will also learn about research-based strategies that help students come to know mathematics and be confident in their ability to do mathematics. These two things—your knowledge of mathematics and how students learn mathematics—are the most important tools you can acquire to be successful.

Becoming an Effective Teacher of Mathematics

As part of your personal desire to build successful learners of mathematics, you might recognize the challenge that mathematics is sometimes seen as the subject that people love to hate. At social events of all kinds—even at parent–teacher conferences—other adults may respond to the fact that you are a teacher of mathematics with comments such as "I could never do math," or "I can't calculate the tip at a restaurant–I just hope they include suggestions for tips at the bottom of my receipt." Instead of dismissing or ignoring these disclosures, consider what positive action you can take. Would people confide that they don't read and hadn't read a book in years—not likely. Families' and teachers' attitudes toward mathematics may enhance or detract from students' ability to do math. It is important for you and for students' families to know that mathematics ability is not inherited—anyone can learn mathematics. Moreover, learning mathematics is an essential life skill (OECD, 2016). So, you need to find ways of countering negative statements about mathematics, especially if they are declared in the presence of students. Point out that it is a myth that only some people can be successful in learning mathematics. Only in that way can the chain of passing apprehension from family member to child, or in rare cases from teacher to student, be broken. There is much joy to be had in solving mathematical problems, and it is essential that you model an excitement for learning and nurture a passion for mathematics in your students.

Ultimately, your students need to think of themselves as mathematicians in the same way as they think of themselves as readers. As students interact with our increasingly mathematical and technological world, they need to construct, modify, communicate or integrate new information in many forms. Solving novel problems and approaching new situations with a mathematical perspective should come as naturally as using reading to comprehend facts, insights, or news. Particularly because this century is a quantitative one (Hacker, 2016), we must prepare students to interpret the language and power of numeracy. Hacker states that "decimals and ratios are now as crucial as nouns and verbs" (p. 2). So, for your students' sake, consider how important mathematics is to interpreting and successfully surviving in our complex economy and in our changing environment. Learning mathematics opens up a world of important ideas to students.

The goal of this book is to help you understand the mathematics methods that will make you an effective teacher. We also base this book on the collective wisdom of an organization of mathematics educators and mathematicians who developed a set of standards for what knowledge, skills and dispositions are important in cultivating a well-prepared beginning teacher of mathematics (Association of Mathematics Teacher Educators, 2017). This book infuses those standards for developing elementary and middle school teachers of mathematics using the suggestions of what best supports teacher candidates in methods courses. Because the authors of this book were also engaged in the creation and writing of the *Standards for Preparing Teachers of Mathematics*, the book is aligned with the AMTE standards. As you dig into the information in the chapters ahead, your vision of what is possible for all students and your confidence to explore and teach mathematics will grow.

A Changing World

In *The World Is Flat* (2007), Thomas Friedman discusses how globalization has created the need for people to have skills that are long lasting and will survive the ever-changing landscape of available jobs. He names categories of workers who regardless of the shifting terrain of job options—will always be successful in finding employment. One of these "untouchable" categories is—math lover. Friedman emphasizes that in a world that is digitized and surrounded by algorithms, math lovers will always have career opportunities and choices. Yet, there is a skills gap of qualified people as science, technology, engineering, and mathematics (STEM) jobs take more than twice as long to fill as other jobs in the marketplace (Rothwell, 2014).

Now every teacher of mathematics has the job to prepare students with career skills while developing a "love of math" in students. Lynn Arthur Steen, a well-known mathematician and educator, stated, "As information becomes ever more quantitative and as society relies increasingly on computers and the data they produce, an innumerate citizen today is as vulnerable as the illiterate peasant of Gutenberg's time" (1997, p. xv). So, as you see there are an array of powerful reasons why children will benefit from the study of mathematics and the instructional approaches you will learn in this book. Your students need to acquire the mental tools to make sense of mathematics—in some cases for mathematical applications that might not yet be known! This knowledge serves as a lens for interpreting the world.

Our changing world influences what should be taught in preK–8 mathematics classrooms as there is a relationship between early mathematics performance and success in middle school (Bailey, Siegler, & Geary, 2014) and high school mathematics (Watts, Duncan, Siegler, & Davis-Kean, 2014). As we prepare preK–8 students for jobs that possibly do not currently exist, we can predict that there will be few jobs where just knowing simple computation is enough to be successful. We can also predict that many jobs will require interpreting complex data, designing algorithms to make predictions, and using multiple strategies to approach new problems.

As you prepare to help students learn mathematics for the future, you will need some perspective on the forces that effect change in the mathematics classroom. This chapter addresses the leadership that you, the teacher, will develop as you shape the mathematics experience for your students. Your beliefs about what it means to know and do mathematics and about how students make sense of mathematics will affect how you approach instruction and the understandings and skills your students take from the classroom. The enthusiasm you demonstrate about mathematical ideas will translate into your students' interest in this amazing and beautiful discipline.

Factors to Consider

Over the years, there have been significant reforms in mathematics education that reflect the technological and informational needs of our society, research on how students learn mathematics, the importance of providing opportunities to learn for all students, and ideas on how and what to teach from an international perspective. Just as we would not expect doctors to be using the exact same techniques and medicines that were prevalent when you were a child, teachers' methods are evolving and transforming via a powerful collection of expert knowledge about how the mind functions and how to design effective instruction (Wiggins, 2013).

There are several significant factors in this transformation. One factor is the public or political pressure for change in mathematics education due largely to information about student performance in national and international studies. These large-scale comparisons of student performance continue to make headlines, provoke public opinion, and pressure legislatures to call for tougher standards backed by testing. This research is important because international and national assessments provide strong evidence that mathematics teaching *must* change if students are to be competitive in the global market and able to understand the complex issues they will confront as responsible citizens of the world (Green, 2014).

National Assessment of Education Progress (NAEP). Since the 1960s, the United States regularly gathers data on how fourth-, eighth-, and twelfth-grade students are doing in mathematics on the NAEP (https://nces.ed.gov/nationsreportcard). These data provide a tool for policy makers and educators to measure the overall improvement of U.S. students over time in what is called the "Nation's Report Card." NAEP uses four achievement levels: below basic, basic, proficient, and advanced, with proficient and advanced representing substantial grade-level achievement. The criterion-referenced test is designed to reflect the current curriculum but keeps a few stable items for purposes of long-term comparison. In the most recent NAEP mathematics assessment in 2015, less than half of all U.S. students in grades 4 and 8 performed at the desirable levels of proficient and advanced (40 percent in fourth grade and 33 percent in eighth grade) (National Center for Education Statistics, 2015). Despite encouraging gains in the NAEP scores over the last 30 years due to important shifts in instructional practices (particularly at the elementary level) (Kloosterman, Rutledge, & Kenney, 2009b), students' performance in 2015 still reveals disappointing levels of competency. For the first time in 25 years the number of students performing at proficient and advanced dropped two points at fourth grade and three points at eighth grade (Toppo, 2015). We still have work to do!

Trends in International Mathematics and Science Study (TIMSS). In 2015, 49 nations participated in the third International Mathematics and Science Study (https://timssandpirls.bc.edu), the largest international comparative study of students' mathematics and science achievement—given regularly since 1995. Data are gathered in grades 4, and 8 from a randomly selected group of students resulting in a sample of more than 600,000 with approximately 20,000 of the students from the United States. The results revealed that U.S. students performed above the international average of the TIMSS countries at both the fourth grade and the eighth grade but were outperformed at the fourth-grade level by education systems in Singapore, Hong Kong, Republic of Korea, Chinese Taipei, Japan, Northern Ireland, Russian Federation, Norway, Ireland, England, Belgium, Kazakhstan, and Portugal and outperformed at the eighth-grade level by education systems in Singapore, Republic of Korea, Chinese Taipei, Hong Kong, Japan, Russian Federation, Kazakhstan, Canada, and Ireland. These data provide valuable benchmarks that allow the United States to reflect on our teaching practices and our overall competitiveness in preparing students for a global economy. If you've heard people talk about how mathematics is taught in Singapore—these rankings are why. But these data do not suggest that we should use the curriculum from other higher performing countries as there are many variables to consider. However we can learn a common theme from these examples: a teaching focus in these nations that emphasizes conceptual understanding and procedural fluency. Both of which are critically important to the long-term growth of problem solving skills (OECD, 2016; Rittle-Johnson, Schenider, & Star, 2015). In fact, teaching in the high-achieving countries more closely resembles the long-standing recommendations of the National Council of Teachers of Mathematics, the major professional organization for mathematics teachers, discussed next.

National Council of Teachers of Mathematics (NCTM). One transformative factor in the teaching of mathematics is the leadership of the National Council of Teachers of Mathematics (NCTM). The NCTM, with more than 60,000 members, is the world's largest mathematics education organization. This group holds an influential role in the support of teachers and an emphasis on what is best for learners. Their guidance in the creation and dissemination of standards for curriculum, assessment, and teaching led the way for other disciplines to create standards and for the eventual creation of the CCSS-M. For an array of resources, including the web-based Illuminations component which consists of a set of exciting instructional experiences for your students, visit the NCTM website (www.nctm.org).

MyLab Education **Self-Check 1.1**

The Movement toward Shared Standards

We share the history of the standards here so you have a sense of how mathematics instruction has changed over time and how external factors and emerging research play a role in that process. These important ideas are all connected to your future as a teacher of elementary or middle school mathematics.

The momentum for reform in mathematics education began in earnest in the early 1980s. The main impetus was a response to a need for more problem solving as well as the research of developmental psychologists who identified how students can best learn mathematics. Then in 1989, NCTM published the first set of standards for a subject area in the *Curriculum and Evaluation Standards for School Mathematics*. Many believe that no other document has had such an enormous effect on school mathematics or on any other area of the curriculum.

NCTM followed in 1991 with a set of standards for teaching that articulated a vision of teaching mathematics for all students, not just a few. In 1995, NCTM added to the collection the *Assessment Standards for School Mathematics*, which focused on the importance of integrating assessment with instruction and indicated the key role that assessment plays in implementing change (see Chapter 5). In 2000, NCTM released *Principles and Standards for School Mathematics* as an update of its original standards document. Combined, these documents prompted a revolutionary reform movement in mathematics education throughout the world.

As these documents influenced teacher practice, ongoing debate about the mathematics curriculum continued with many arguing that instead of hurrying through numerous topics every year, the curriculum needed to address content more deeply. Guidance was needed in deciding what mathematics content should be taught at each grade level and, in 2006, NCTM released *Curriculum Focal Points*, a little publication with a big message—the mathematics taught at each grade level needs to be focused, provide depth, and explicitly show connections. The goal of the Focal Points was to support a coherent curriculum and give clarity to teachers and students as to what should be taught at each grade. The resulting sequence of key concepts provided a "structural fiber" that helped students understand mathematics (Dossey, McCrone, & Halvorsen, 2016, p. 18).

In 2010, the National Governors Association (NGA) Center for Best Practices and Council of Chief State School Officers (CCSSO) presented the *Common Core State Standards*, which are grade-level specific standards which incorporate ideas from *Curriculum*

Focal Points as well as international curriculum documents. A large majority of U.S. states adopted these as their standards and other states were stimulated to create new standards of their own. In less than 25 years, the standards movement transformed the country from having little to no coherent vision on what mathematics should be taught and when, to a more widely shared idea of what students should know and be able to do at each grade level.

In the following sections, we discuss three significant components of the standards that are critical to your work as a highly effective teacher of mathematics.

Mathematics Content Standards

As noted earlier, the dialogue on improving mathematics teaching and learning extends beyond mathematics educators. Policymakers and elected officials considered previous NCTM standards documents, international assessments, and research on the best way to prepare students to be "college and career ready." The National Governors Association Center for Best Practices and the Council of Chief State School Officers (CCSSO) collaborated with other professional groups and entities to develop shared expectations for K–12 students across states, a focused set of mathematics content standards and practices, and efficiency of material and assessment development (Porter, McMaken, Hwang, & Yang, 2011). As a result, they developed *Common Core State Standards for Mathematics* (CCSS-M) which can be downloaded at http://www.corestandards.org/math.

The CCSS-M articulates an overview of *critical areas* of mathematics content that are expectations for each grade from K–8 to provide a coherent curriculum built around big mathematical ideas. These larger groups of related standards are called *domains,* and there are eleven that relate to grades K–8 (see Figure 1.1). At this time, approximately 37 states, Washington, D.C., four territories, and Department of Defense Schools have adopted the CCSS-M. A few states chose not to adopt the standards from the start, some created their own versions, and others are still deciding their level of participation or reevaluating their own standards compared to CCSS-M. This change represents the largest shift of mathematics content in the United States in more than 100 years.

MyLab Education **Video Example 1.1**

Watch this video (https://www.youtube.com/watch?v=5pBOnvzC_Yw&list=PLD7F4C7DE7CB3D2E6&index=15) by one of the authors of the CCSS-M to hear more about the shifts made in these standards.

The *Common Core State Standards* were developed with strong consideration given to the research on what is known about the development of students' understanding of mathematics over time (Cobb & Jackson, 2011). The selection of topics at particular grades reflects not only rigorous mathematics, but also what is known from research and practice about learning progressions which are sometimes referred to as *learning trajectories* (Clements & Sarama, 2014; Confrey, Maloney, & Corley, 2014). These progressions can help teachers know the sequence of what came before a particular concept as well as what to expect next as students reach key points

<table>
<tr><th>Kindergarten</th><th>Grade 1</th><th>Grade 2</th><th>Grade 3</th><th>Grade 4</th><th>Grade 5</th><th>Grade 6</th><th>Grade 7</th><th>Grade 8</th></tr>
<tr><td>Counting and Cardinality</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td colspan="6">Operations and Algebraic Thinking</td><td colspan="3">Expressions and Equations</td></tr>
<tr><td colspan="6">Number and Operations in Base-Ten</td><td colspan="3">The Number System</td></tr>
<tr><td colspan="6">Measurement and Data</td><td colspan="3">Statistics and Probability</td></tr>
<tr><td colspan="9">Geometry</td></tr>
<tr><td></td><td></td><td></td><td colspan="3">Number and Operations—Fractions</td><td colspan="2">Ratios and Proportional Relationships</td><td>Functions</td></tr>
</table>

FIGURE 1.1 *Common Core State Standards* domains by grade level.

along a pathway to desired learning targets (Corcoran, Mosher, & Rogat, 2009). Although these paths are not identical for all students, they can inform the order of instructional experiences which will support movement toward understanding and application of mathematics concepts. There is a website for the "Progressions Documents for the Common Core Math Standards" (http://ime.math.arizona.edu/progressions) where progressions for the domains in the Common Core State Standards can be found.

Although you may have heard people suggest that they are not in favor of the *Common Core State Standards*, many of those comments reflect people's concern with the testing that is associated with the standards, not the content standards or the mathematical practices which are described next.

The Process Standards and Standards for Mathematical Practice

To prepare students for college and career readiness and a lifetime of enjoying mathematical ideas, there are additional standards that emphasize the important processes in doing mathematics. The process standards refer to the mathematical methods and strategies which preK–12 students acquire to enhance their use of mathematical content knowledge. NCTM developed these standards as part of the *Principles and Standards* document (2000) and stated that the process standards should not be regarded as separate content or strands in the mathematics curriculum, rather, they are central and integral components of all mathematics learning and teaching. The five process standards and ways you can develop these elements in your students can be found in Table 1.1. Members of NCTM have free access to the *Principles and Standards* and nonmembers can sign up for 120 days of free access to the full document on the NCTM website (www.nctm.org) under the tab *Standards and Focal Points.*

The *Common Core State Standards* also go beyond specifying mathematics content expectations to also include Standards for Mathematical Practice. These are "'processes and proficiencies'

TABLE 1.1 THE FIVE PROCESS STANDARDS FROM *PRINCIPLES AND STANDARDS FOR SCHOOL MATHEMATICS*

Process Standard	How Can You Develop These Processes in Your Students?
Problem Solving	• Start instruction with a problem to solve—as problem solving is the vehicle for developing mathematical ideas. • Select meaningful mathematical tasks. • Set problems in a situation to which students can relate. • Use a variety of strategies to solve problems. • Have students self-assess their understanding of the problem and their strategy use.
Reasoning and Proof	• Have students consider evidence of why something is true or not. • Create opportunities for students to evaluate conjectures—do they hold true? • Encourage students to use logical reasoning to see if something always works or their answers make sense. • Demonstrate a variety of ways for students to justify their thinking through finding examples and counter-examples to use in a logical argument.
Communication	• Invite students to talk about, write about, describe, and explain their mathematical ideas as a way to examine their thinking. • Give students opportunities to share ideas so that others can understand and actively discuss their reasoning. • Share examples of student work, so students can compare and assess others' thinking. • Present precise mathematical language and notation so that the word usage and definitions can act as a foundation for students' future learning.
Connections	• Emphasize how mathematical ideas explicitly connect to students' prior mathematical knowledge and future learning. • Assist students in developing the relationships between the mathematics being learned and real-world contexts and in other subject areas.
Representation	• Encourage students to use multiple representations to explore relationships and communicate their thinking. • Create opportunities for students to move from one representation of a mathematical concept or idea to another to add depth of understanding. • Provide problems where students can use mathematical models to clarify or represent a situation.

Source: Adapted with permission from NCTM (National Council of Teachers of Mathematics). (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.